

WEIGHTED LIPSCHITZ CONTINUITY, SCHWARZ-PICK'S LEMMA AND LANDAU-BLOCH'S THEOREM FOR HYPERBOLIC-HARMONIC MAPPINGS IN \mathbb{C}^n

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ABSTRACT. In this paper, we discuss some properties on hyperbolic-harmonic mappings in the unit ball of \mathbb{C}^n . First, we investigate the relationship between the weighted Lipschitz functions and the hyperbolic-harmonic Bloch spaces. Then we establish the Schwarz-Pick type theorem for hyperbolic-harmonic mappings and apply it to prove the existence of Landau-Bloch constant for mappings in α -Bloch spaces.

1. INTRODUCTION AND PRELIMINARIES

Let \mathbb{C}^n denote the complex Euclidean n -space. For $z = (z_1, \dots, z_n) \in \mathbb{C}^n$, the conjugate of z , denoted by \bar{z} , is defined by $\bar{z} = (\bar{z}_1, \dots, \bar{z}_n)$. For z and $w = (w_1, \dots, w_n) \in \mathbb{C}^n$, the standard Hermitian scalar product on \mathbb{C}^n and the Euclidean norm of z are given by

$$\langle z, w \rangle := \sum_{k=1}^n z_k \bar{w}_k \quad \text{and} \quad |z| := \langle z, z \rangle^{1/2} = (|z_1|^2 + \dots + |z_n|^2)^{1/2},$$

respectively. For $a \in \mathbb{C}^n$, $\mathbb{B}^n(a, r) = \{z \in \mathbb{C}^n : |z - a| < r\}$ is the (open) ball of radius r with center a . Also, we let $\mathbb{B}^n(r) := \mathbb{B}^n(0, r)$ and use \mathbb{B}^n to denote the unit ball $\mathbb{B}^n(1)$, and $\mathbb{D} = \mathbb{B}^1$. We can interpret \mathbb{C}^n as the real $2n$ -space \mathbb{R}^{2n} so that a ball in \mathbb{C}^n is also a ball in \mathbb{R}^{2n} . Following the standard convention, for $a \in \mathbb{R}^n$, we may let $\mathbb{B}_{\mathbb{R}}^n(a, r) = \{x \in \mathbb{R}^n : |x - a| < r\}$ so that $\mathbb{B}_{\mathbb{R}}^n(r) := \mathbb{B}_{\mathbb{R}}^n(0, r)$ and $\mathbb{B}_{\mathbb{R}}^n = \mathbb{B}_{\mathbb{R}}^n(1)$ denotes the open unit ball in \mathbb{R}^n centered at the origin.

Definition 1. A twice continuously differentiable complex-valued function $f = u + iv$ on \mathbb{B}^n is called a *hyperbolic-harmonic* (briefly, *h-harmonic*, in the following) if and only if the real-valued functions u and v satisfy $\Delta_h u = \Delta_h v = 0$ on \mathbb{B}^n , where

$$\Delta_h := (1 - |z|^2)^2 \sum_{k=1}^n \left(\frac{\partial}{\partial x_k^2} + \frac{\partial}{\partial y_k^2} \right) + 4(n-1)(1 - |z|^2) \sum_{k=1}^n \left(x_k \frac{\partial}{\partial x_k} + y_k \frac{\partial}{\partial y_k} \right)$$

denotes the *Laplace-Beltrami operator* and $z_k = x_k + iy_k$ for $k = 1, \dots, n$.

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Obviously, when $n = 1$, all h -harmonic mappings are planar harmonic mappings. We refer to [4, 11, 12, 24] for more details of h -harmonic mappings.

It turns out that if $\psi \in C(\partial\mathbb{B}^n)$, then the Dirichlet problem

$$\begin{cases} \Delta_h f = 0 & \text{in } \mathbb{B}^n \\ f = \psi & \text{on } \partial\mathbb{B}^n \end{cases}$$

has unique solution in $C(\overline{\mathbb{B}^n})$ and can be represented by

$$f(z) = \int_{\partial\mathbb{B}^n} P_h(z, \zeta) \psi(\zeta) d\sigma(\zeta),$$

where $d\sigma$ is the unique normalized surface measure on $\partial\mathbb{B}^n$ and $P_h(z, \zeta)$ is the *hyperbolic Poisson kernel* defined by

$$P_h(z, \zeta) = \left(\frac{1 - |z|^2}{|z - \zeta|^2} \right)^{2n-1} \quad (z \in \mathbb{B}^n, \zeta \in \partial\mathbb{B}^n).$$

Here $C(\Omega)$ stands for the set of all continuous functions on Ω . A planar harmonic mapping f in \mathbb{D} is called a *harmonic Bloch mapping* if and only if

$$\beta_f = \sup_{z, w \in \mathbb{D}, z \neq w} \frac{|f(z) - f(w)|}{\rho(z, w)} < \infty.$$

Here β_f is the *Lipschitz number* of f and

$$\rho(z, w) = \frac{1}{2} \log \left(\frac{1 + \left| \frac{z-w}{1-\bar{z}w} \right|}{1 - \left| \frac{z-w}{1-\bar{z}w} \right|} \right) = \operatorname{arctanh} \left| \frac{z-w}{1-\bar{z}w} \right|$$

denotes the hyperbolic distance between z and w in \mathbb{D} . It can be proved that

$$\beta_f = \sup_{z \in \mathbb{D}} \left\{ (1 - |z|^2) [|f_z(z)| + |f_{\bar{z}}(z)|] \right\}.$$

We refer to [10, Theroem 2] (see also [7, 8]) for a proof of the last fact.

For a complex-valued h -harmonic mapping f on \mathbb{B}^n , we introduce

$$\widehat{\nabla} f = \left(\frac{\partial f}{\partial z_1}, \dots, \frac{\partial f}{\partial z_n} \right) \quad \text{and} \quad \widehat{\nabla} \bar{f} = \left(\frac{\partial f}{\partial \bar{z}_1}, \dots, \frac{\partial f}{\partial \bar{z}_n} \right).$$

Definition 2. The *h -harmonic Bloch space* \mathcal{HB} consists of complex-valued h -harmonic mappings f defined on \mathbb{B}^n such that

$$\|f\|_{\mathcal{HB}} = \sup_{z \in \mathbb{B}^n} \left\{ (1 - |z|^2) [|\widehat{\nabla} f(z)| + |\widehat{\nabla} \bar{f}(z)|] \right\} < \infty.$$

Obviously, when $n = 1$, $\|f\|_{\mathcal{HB}} = \beta_f$. For a pair of distinct points z and w in \mathbb{B}^n , let

$$\mathcal{L}_f(z, w) = \frac{(1 - |z|^2)^{\frac{1}{2}} (1 - |w|^2)^{\frac{1}{2}} |f(z) - f(w)|}{|z - w|}$$

denote the *weighted Lipschitz function* of a given h -harmonic mapping $f : \mathbb{B}^n \rightarrow \mathbb{C}$. The relationship between weighted Lipschitz functions and (analytic) Bloch spaces has attracted much attention (cf. [1, 10, 13, 14, 17, 20]). Our first aim is to characterize the mappings in h -harmonic Bloch spaces in terms of their corresponding

weighted Lipschitz functions. This is done in Theorem 1 which is indeed a generalization of [10, Theorem 1] and [13, Theorem 3].

Throughout, $\mathcal{H}(\mathbb{B}^n, \mathbb{C}^n)$ denotes the set of all continuously differentiable mappings f from \mathbb{B}^n into \mathbb{C}^n with $f = (f_1, \dots, f_n)$ and $f_i(z) = u_i(z) + iv_i(z)$, where u_i and v_i are real-valued mappings on \mathbb{B}^n . For $f \in \mathcal{H}(\mathbb{B}^n, \mathbb{C}^n)$, the real Jacobian matrix of f is given by

$$J_f = \begin{pmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial y_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial y_2} & \dots & \frac{\partial u_1}{\partial x_n} & \frac{\partial u_1}{\partial y_n} \\ \frac{\partial v_1}{\partial x_1} & \frac{\partial v_1}{\partial y_1} & \frac{\partial v_1}{\partial x_2} & \frac{\partial v_1}{\partial y_2} & \dots & \frac{\partial v_1}{\partial x_n} & \frac{\partial v_1}{\partial y_n} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial y_1} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_2}{\partial y_2} & \dots & \frac{\partial u_2}{\partial x_n} & \frac{\partial u_2}{\partial y_n} \\ \frac{\partial v_2}{\partial x_1} & \frac{\partial v_2}{\partial y_1} & \frac{\partial v_2}{\partial x_2} & \frac{\partial v_2}{\partial y_2} & \dots & \frac{\partial v_2}{\partial x_n} & \frac{\partial v_2}{\partial y_n} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{\partial u_n}{\partial x_1} & \frac{\partial u_n}{\partial y_1} & \frac{\partial u_n}{\partial x_2} & \frac{\partial u_n}{\partial y_2} & \dots & \frac{\partial u_n}{\partial x_n} & \frac{\partial u_n}{\partial y_n} \\ \frac{\partial v_n}{\partial x_1} & \frac{\partial v_n}{\partial y_1} & \frac{\partial v_n}{\partial x_2} & \frac{\partial v_n}{\partial y_2} & \dots & \frac{\partial v_n}{\partial x_n} & \frac{\partial v_n}{\partial y_n} \end{pmatrix}.$$

A vector-valued mapping $f \in \mathcal{H}(\mathbb{B}^n, \mathbb{C}^n)$ is said to be *h-harmonic*, if each component f_i ($1 \leq i \leq n$) is *h-harmonic* mapping from \mathbb{B}^n into \mathbb{C} . We denote by $\mathcal{H}_h(\mathbb{B}^n, \mathbb{C}^n)$ the set of all vector-valued *h-harmonic* mappings from \mathbb{B}^n into \mathbb{C}^n .

For each $f = (f_1, \dots, f_n) \in \mathcal{H}(\mathbb{B}^n, \mathbb{C}^n)$, denote by

$$f_z = (\widehat{\nabla} f_1, \dots, \widehat{\nabla} f_n)^T$$

the matrix formed by the complex gradients $\widehat{\nabla} f_1, \dots, \widehat{\nabla} f_n$, and let

$$f_{\bar{z}} = (\widehat{\nabla} \bar{f}_1, \dots, \widehat{\nabla} \bar{f}_n)^T,$$

where T means the matrix transpose.

For an $n \times n$ matrix $A = (a_{ij})_{n \times n}$, the operator norm of A is given by

$$|A| = \sup_{z \neq 0} \frac{|Az|}{|z|} = \max \{ |A\theta| : \theta \in \partial \mathbb{B}^n \}.$$

Then for $f \in \mathcal{H}(\mathbb{B}^n, \mathbb{C}^n)$, we use the standard notations:

$$(1) \quad \Lambda_f(z) = \max_{\theta \in \partial \mathbb{B}^n} |f_z(z)\theta + f_{\bar{z}}(z)\bar{\theta}| \quad \text{and} \quad \lambda_f(z) = \min_{\theta \in \partial \mathbb{B}^n} |f_z(z)\theta + f_{\bar{z}}(z)\bar{\theta}|.$$

We see that (see for instance [5])

$$(2) \quad \Lambda_f = \max_{\theta \in \partial \mathbb{B}_{\mathbb{R}}^{2n}} |J_f \theta| \quad \text{and} \quad \lambda_f = \min_{\theta \in \partial \mathbb{B}_{\mathbb{R}}^{2n}} |J_f \theta|.$$

Let $\mathcal{PH}(\mathbb{B}^n, \mathbb{C}^n)$ denote the set of all $f = (f_1, \dots, f_n) \in \mathcal{H}(\mathbb{B}^n, \mathbb{C}^n)$ such that all partial derivatives $\partial f_j / \partial z_k$ and $\partial f_j / \partial \bar{z}_k$ ($1 \leq j, k \leq n$) are *h-harmonic* in \mathbb{B}^n .

We remark that when $n = 1$, every harmonic mapping from \mathbb{D} to \mathbb{C} belongs to $\mathcal{PH}(\mathbb{D}, \mathbb{C})$. The converse is not true as the function $f(z) = |z|^2$ shows.

Definition 3. For $\alpha > 0$, the *vector-valued h-harmonic α -Bloch space* $\mathcal{HB}_n(\alpha)$ consists of all mappings in $\mathcal{PH}(\mathbb{B}^n, \mathbb{C}^n)$ such that

$$\|f\|_{\mathcal{HB}_n(\alpha)} = \sup_{z \in \mathbb{B}^n} \{ (1 - |z|^2)^\alpha [|f_z(z)| + |f_{\bar{z}}(z)|] \} < \infty.$$

Obviously, $\mathcal{HB}_1(\alpha)$ contains the harmonic α -Bloch space as a proper subset (see [8]). One of the long standing open problems in function theory is to determine the precise value of the univalent Landau-Bloch constant for analytic functions of \mathbb{D} . In recent years, this problem has attracted much attention, see [3, 16, 18] and references therein. For general holomorphic mappings of more than one complex variable, no Landau-Bloch constant exists (cf. [25]). In order to obtain some analogs of Landau-Bloch's theorem for mappings with several complex variables, it became necessary to restrict the class of mappings considered (cf. [2, 5, 9, 15, 21, 23, 25]).

Based on Heinz's Lemma and Colonna's distortion theorem ([10, Theorem 3]) for planar harmonic mappings, in [5], the authors established the Schwarz-Pick type theorem for bounded pluriharmonic mappings and pluriharmonic K -mappings. As a consequence of it, the authors in [5] obtained Landau-Bloch theorem as generalizations of the main results from [6]. It is known that every pluriharmonic mapping f defined in \mathbb{B}^n admits a decomposition $f = h + \bar{g}$, where h and g are holomorphic in \mathbb{B}^n . This decomposition property is no longer valid for mappings in $\mathcal{HB}_n(\alpha)$. Hence the methods of proof used in [5] are no longer applicable for mappings in $\mathcal{H}_h(\mathbb{B}^n, \mathbb{C}^n)$ and $\mathcal{HB}_n(\alpha)$. In view of this reasoning, in this article, we use entirely a different approach and prove Schwarz-Pick type theorem for mappings in $\mathcal{H}_h(\mathbb{B}^n, \mathbb{C}^n)$ and then establish the Landau-Bloch theorem for mappings in $\mathcal{HB}_n(\alpha)$ (see Theorems 2 and 3). It is worth pointing out that Theorems 2 and 3 are indeed generalizations of [10, Theorem 1] and [8, Theorem 2.4], respectively.

2. CHARACTERIZATION OF MAPPINGS IN h -HARMONIC BLOCH SPACES

Consider the group $\text{Aut}(\mathbb{B}^n)$ consisting of all biholomorphic mappings of \mathbb{B}^n onto itself. Then for each $a \in \mathbb{B}^n$, ϕ_a defined by [22]:

$$\phi_a(z) = \frac{a - P_a z - (1 - |a|^2)^{\frac{1}{2}}(z - P_a z)}{1 - \langle z, a \rangle}$$

belongs to $\text{Aut}(\mathbb{B}^n)$, where $P_a z = \frac{a\langle z, a \rangle}{\langle a, a \rangle}$. Moreover, we find that

$$(3) \quad 1 - |\phi_a(z)|^2 = \frac{(1 - |z|^2)(1 - |a|^2)}{|1 - \langle z, a \rangle|^2}.$$

Using the arguments as in the proof of [19, Lemma 2.5], we have

Lemma 1. *Suppose $f : \overline{\mathbb{B}}_{\mathbb{R}}^n(a, r) \rightarrow \mathbb{R}$ is a continuous, and h -harmonic in $\mathbb{B}_{\mathbb{R}}^n(a, r)$. Then*

$$|\nabla f(a)| \leq \frac{2(n-1)\sqrt{n}}{nV(n)r^n} \int_{\partial \mathbb{B}_{\mathbb{R}}^n(a, r)} |f(a) - f(t)| d\sigma(t),$$

where $d\sigma$ denotes the surface measure on $\partial \mathbb{B}_{\mathbb{R}}^n(a, r)$ and $V(n)$, the volume of the unit ball in \mathbb{R}^n .

Proof. Without loss of generality, we may assume that $a = 0$ and $f(0) = 0$. Let

$$K(x, t) = \frac{1}{nr^{n-1}V(n)} \left(\frac{r^2 - |x|^2}{|x - t|^2} \right)^{n-1}.$$

Then by the assumption on f , we see that [4]

$$f(x) = \int_{\partial \mathbb{B}_{\mathbb{R}}^n(r)} K(x, t) f(t) d\sigma(t), \quad x \in \mathbb{B}_{\mathbb{R}}^n(r).$$

Further, a computation shows that

$$\frac{\partial}{\partial x_i} K(x, t) = \frac{-2(n-1)(r^2 - |x|^2)^{n-2}}{nr^{n-1}V(n)} \cdot \frac{[|x-t|^2 x_i + (r^2 - |x|^2)(x_i - t_i)]}{|x-t|^{2n}}$$

which yields

$$\frac{\partial}{\partial x_i} K(0, t) = \frac{2(n-1)t_i}{nV(n)r^{n+1}}$$

whence

$$\begin{aligned} |\nabla f(0)| &= \left[\sum_{i=1}^n \left| \int_{\partial \mathbb{B}_{\mathbb{R}}^n(r)} \frac{\partial}{\partial x_i} K(0, t) f(t) d\sigma(t) \right|^2 \right]^{\frac{1}{2}} \\ &\leq \sum_{i=1}^n \left| \int_{\partial \mathbb{B}_{\mathbb{R}}^n(r)} \frac{\partial}{\partial x_i} K(0, t) f(t) d\sigma(t) \right| \\ &\leq \int_{\partial \mathbb{B}_{\mathbb{R}}^n(r)} |f(t)| \sum_{i=1}^n \left| \frac{\partial}{\partial x_i} K(0, t) \right| d\sigma(t) \\ &\leq \sqrt{n} \int_{\partial \mathbb{B}_{\mathbb{R}}^n(r)} |f(t)| \left(\sum_{i=1}^n \left| \frac{\partial}{\partial x_i} K(0, t) \right|^2 \right)^{\frac{1}{2}} d\sigma(t) \\ &= \frac{2(n-1)\sqrt{n}}{nV(n)r^n} \int_{\partial \mathbb{B}_{\mathbb{R}}^n(r)} |f(t)| d\sigma(t), \end{aligned}$$

from which the lemma follows. \square

Lemma 2. *Let $f = u + iv$ be a continuously differentiable mapping from \mathbb{B}^n into \mathbb{C} , where u and v are real-valued functions. Then for $z \in \mathbb{B}^n$,*

$$(4) \quad |\widehat{\nabla} f(z)| + |\widehat{\nabla} \bar{f}(z)| \leq |\nabla u(z)| + |\nabla v(z)|,$$

where $\nabla u = \left(\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial y_1}, \dots, \frac{\partial u}{\partial x_n}, \frac{\partial u}{\partial y_n} \right)$ and $\nabla v = \left(\frac{\partial v}{\partial x_1}, \frac{\partial v}{\partial y_1}, \dots, \frac{\partial v}{\partial x_n}, \frac{\partial v}{\partial y_n} \right)$.

Proof. By the basic change of variables, for each $k = 1, 2, \dots, n$, we have

$$f_{z_k}(z) = \frac{1}{2}(f_{x_k}(z) - if_{y_k}(z)) \text{ and } f_{\bar{z}_k}(z) = \frac{1}{2}(f_{x_k}(z) + if_{y_k}(z))$$

which implies

$$f_{z_k}(z) = \frac{1}{2}[u_{x_k}(z) + v_{y_k}(z) + i(v_{x_k}(z) - u_{y_k}(z))]$$

and similarly,

$$f_{\bar{z}_k}(z) = \frac{1}{2}[u_{x_k}(z) - v_{y_k}(z) + i(v_{x_k}(z) + u_{y_k}(z))].$$

The classical Cauchy-Schwarz inequality gives

$$|\widehat{\nabla} f(z)| = \frac{1}{2} \sqrt{\sum_{k=1}^n \left[(u_{x_k}(z) + v_{y_k}(z))^2 + (v_{x_k}(z) - u_{y_k}(z))^2 \right]} \leq \frac{1}{2} (|\nabla u(z)| + |\nabla v(z)|)$$

and similarly,

$$|\widehat{\nabla} \bar{f}(z)| = \frac{1}{2} \sqrt{\sum_{k=1}^n \left[(u_{x_k}(z) - v_{y_k}(z))^2 + (v_{x_k}(z) + u_{y_k}(z))^2 \right]} \leq \frac{1}{2} (|\nabla u(z)| + |\nabla v(z)|)$$

from which we obtain the desired inequality (4). \square

Example 1. Consider $f(z) = z^2 + \bar{z} = u(x, y) + iv(x, y)$ so that $u(x, y) = x^2 + x - y^2$ and $v(x, y) = 2xy - y$. It is easy to see that

$$|f_z(0)| + |f_{\bar{z}}(0)| = 1 \quad \text{and} \quad |\nabla u(0)| + |\nabla v(0)| = 2$$

showing that strict inequality in (4) is possible.

Theorem 1. $f \in \mathcal{HB}$ if and only if $\sup_{z, w \in \mathbb{B}^n, z \neq w} \mathcal{L}_f(z, w) < \infty$.

Proof. First we prove the necessity. For each pair of distinct points z and w in \mathbb{B}^n , we have

$$\begin{aligned} |f(z) - f(w)| &= \left| \int_0^1 \frac{df}{dt}(zt + (1-t)w) dt \right| \\ &= \left| \sum_{k=1}^n (z_k - w_k) \int_0^1 \frac{df}{d\varsigma_k}(zt + (1-t)w) dt \right. \\ &\quad \left. + \sum_{k=1}^n (\bar{z}_k - \bar{w}_k) \int_0^1 \frac{df}{d\bar{\varsigma}_k}(zt + (1-t)w) dt \right| \\ &\leq \sum_{k=1}^n |z_k - w_k| \cdot \left| \int_0^1 \frac{df}{d\varsigma_k}(zt + (1-t)w) dt \right| \\ &\quad + \sum_{k=1}^n |\bar{z}_k - \bar{w}_k| \cdot \left| \int_0^1 \frac{df}{d\bar{\varsigma}_k}(zt + (1-t)w) dt \right|, \end{aligned}$$

where $\varsigma = (\varsigma_1, \dots, \varsigma_n) = zt + (1-t)w$. Hence we see that

$$\begin{aligned} |f(z) - f(w)| &\leq \left(\sum_{k=1}^n |z_k - w_k|^2 \right)^{\frac{1}{2}} \left\{ \left[\sum_{k=1}^n \left(\int_0^1 \left| \frac{\partial f}{\partial \varsigma_k}(zt + (1-t)w) \right| dt \right)^2 \right]^{\frac{1}{2}} \right. \\ &\quad \left. + \left[\sum_{k=1}^n \left(\int_0^1 \left| \frac{\partial f}{\partial \bar{\varsigma}_k}(zt + (1-t)w) \right| dt \right)^2 \right]^{\frac{1}{2}} \right\} \\ &\leq \sqrt{n} |z - w| \int_0^1 \left[|\widehat{\nabla} f(zt + (1-t)w)| + |\widehat{\nabla} \bar{f}(zt + (1-t)w)| \right] dt. \end{aligned}$$

This gives

$$\begin{aligned}
\frac{|f(z) - f(w)|}{|z - w|} &\leq \sqrt{n} \int_0^1 \frac{[|\widehat{\nabla} f(\varsigma)| + |\widehat{\nabla} \bar{f}(\varsigma)|](1 - |\varsigma|^2)}{1 - |\varsigma|^2} dt \\
&\leq \sqrt{n} \|f\|_{\mathcal{HB}} \int_0^1 \frac{dt}{1 - |\varsigma|^2} \\
&\leq \sqrt{n} \|f\|_{\mathcal{HB}} \int_0^1 \frac{dt}{[(1-t)(1-|z|)]^{\frac{1}{2}} [t(1-|w|)]^{\frac{1}{2}}} \\
&= \frac{\pi \sqrt{n} \|f\|_{\mathcal{HB}}}{(1-|z|)^{\frac{1}{2}} (1-|w|)^{\frac{1}{2}}}.
\end{aligned}$$

Thus,

$$\sup_{z, w \in \mathbb{B}^n, z \neq w} \mathcal{L}_f(z, w) \leq \pi \sqrt{n} \|f\|_{\mathcal{HB}}.$$

Next we prove the sufficiency part. Let $f = u + iv$, where u and v are real h -harmonic functions. Fix $r \in (0, 1)$. In view of (3) and the fact that $|\langle z, a \rangle| \leq |z| |a|$, we easily have

$$(5) \quad |\phi_a(z)| \leq \frac{|z - a|}{|1 - \langle z, a \rangle|} \leq \frac{|z - a|}{1 - |a|},$$

whence for $a \in \mathbb{B}^n$,

$$\mathbb{B}^n \left(a, \frac{r(1 - |a|^2)}{2} \right) \subset E(a, r),$$

where $E(a, r) = \{z \in \mathbb{B}^n : |\phi_a(z)| < r\}$. By Lemma 1, we have

$$\begin{aligned}
(1 - |z|^2) |\nabla u(z)| &\leq \frac{(2n-1)\sqrt{2n}(1 - |z|^2)}{nV(2n) \left[\frac{r(1-|z|^2)}{2} \right]^{2n}} \int_{\partial \mathbb{B}^n \left(z, \frac{r(1-|z|^2)}{2} \right)} |u(\zeta) - u(z)| d\sigma(\zeta) \\
&= M(|z|, r) \int_{\partial \mathbb{B}^n \left(z, \frac{r(1-|z|^2)}{2} \right)} |u(\zeta) - u(z)| d\sigma(\zeta),
\end{aligned}$$

where $V(2n)$ denotes the volume of the unit ball in \mathbb{R}^{2n} (or \mathbb{C}^n) and

$$M(|z|, r) = \frac{2^{2n}(2n-1)\sqrt{2n}}{nV(2n)(1 - |z|^2)^{2n-1} r^{2n}}.$$

Similarly, we obtain

$$(1 - |z|^2) |\nabla v(z)| \leq M(|z|, r) \int_{\partial \mathbb{B}^n \left(z, \frac{r(1-|z|^2)}{2} \right)} |v(\zeta) - v(z)| d\sigma(\zeta).$$

By Lemma 2, we have

$$\begin{aligned}
(1 - |z|^2)(|\widehat{\nabla} f(z)| + |\widehat{\nabla} \bar{f}(z)|) &\leq (1 - |z|^2)(|\nabla u(z)| + |\nabla v(z)|) \\
&\leq M(|z|, r) \int_{\partial \mathbb{B}^n \left(z, \frac{r(1-|z|^2)}{2} \right)} \left(|u(\zeta) - u(z)| \right. \\
&\quad \left. + |v(\zeta) - v(z)| \right) d\sigma(\zeta) \\
&\leq \sqrt{2} M(|z|, r) M_1 \int_{\partial \mathbb{B}^n \left(z, \frac{r(1-|z|^2)}{2} \right)} d\sigma(\zeta) \\
&= \frac{4\sqrt{n}(2n-1)}{r} M_1,
\end{aligned}$$

where $M_1 = \sup\{|f(z) - f(w)| : w \in E(z, r)\}$. Hence for all $w \in \mathbb{B}^n \left(z, \frac{r(1-|z|^2)}{2} \right) \subset E(z, r)$, it follows from (3) and (5) that

$$\begin{aligned}
\frac{(1 - |z|^2)^{\frac{1}{2}}(1 - |w|^2)^{\frac{1}{2}}}{|z - w|} &= \frac{(1 - |z|^2)^{\frac{1}{2}}(1 - |w|^2)^{\frac{1}{2}}}{|1 - \langle z, w \rangle|} \cdot \frac{|1 - \langle z, w \rangle|}{|z - w|} \\
&= \sqrt{1 - |\phi_z(w)|^2} \cdot \frac{|1 - \langle z, w \rangle|}{|z - w|} \\
&\geq \sqrt{1 - r^2} \cdot \frac{|1 - \langle z, w \rangle|}{|z - w|} \\
&\geq \frac{\sqrt{1 - r^2}}{r}.
\end{aligned}$$

Therefore, there exists a positive constant $M_2(n, r)$ such that

$$(1 - |z|^2)[|\widehat{\nabla} f(z)| + |\widehat{\nabla} \bar{f}(z)|] \leq M_2(n, r) \sup_{w \in E(z, r), w \neq z} \mathcal{L}_f(z, w),$$

from which we see that $f \in \mathcal{HB}$. □

3. SCHWARZ-PICK TYPE THEOREM AND LANDAU-BLOCH THEOREM

The following result is a Schwarz-Pick type theorem for h -harmonic mappings in $\mathcal{H}_h(\mathbb{B}^n, \mathbb{C}^n)$.

Theorem 2. *Let $f \in \mathcal{H}_h(\mathbb{B}^n, \mathbb{C}^n)$ with $|f(z)| \leq M$ for $z \in \mathbb{B}^n$, where M is a positive constant. Then*

$$(6) \quad \left| f(z) - \frac{(1 - |z|)^{2n-1}}{(1 + |z|)^{2n-1}} f(0) \right| \leq M \left[1 - \frac{(1 - |z|)^{2n-1}}{(1 + |z|)^{2n-1}} \right]$$

and

$$(7) \quad \Lambda_f \leq \frac{2(2n-1)M}{(1 - |z|)^2}.$$

Proof. We first prove (6). Without loss of generality, we assume that f is also h -harmonic on $\partial\mathbb{B}^n$. The hyperbolic Poisson integral formula states that

$$(8) \quad f(z) = \int_{\partial\mathbb{B}^n} P_h(z, \zeta) f(\zeta) d\sigma(\zeta), \quad \int_{\partial\mathbb{B}^n} P_h(z, \zeta) d\sigma(\zeta) = 1.$$

As $P_h(0, \zeta) = 1$ and $P_h(z, \zeta) \leq 1$ for $\zeta \in \partial\mathbb{B}^n$ and all $z \in \mathbb{B}^n$, the representation (8) quickly yields that

$$\begin{aligned} \left| f(z) - \frac{(1 - |z|)^{2n-1}}{(1 + |z|)^{2n-1}} f(0) \right| &= \left| \int_{\partial\mathbb{B}^n} \left[\frac{(1 - |z|^2)^{2n-1}}{|z - \zeta|^{2(2n-1)}} - \frac{(1 - |z|)^{2n-1}}{(1 + |z|)^{2n-1}} \right] f(\zeta) d\sigma(\zeta) \right| \\ &\leq \int_{\partial\mathbb{B}^n} \left[\frac{(1 - |z|^2)^{2n-1}}{|z - \zeta|^{2(2n-1)}} - \frac{(1 - |z|)^{2n-1}}{(1 + |z|)^{2n-1}} \right] |f(\zeta)| d\sigma(\zeta) \\ &\leq M \left[1 - \frac{(1 - |z|)^{2n-1}}{(1 + |z|)^{2n-1}} \right] \end{aligned}$$

and the proof of (6) follows.

Next, we prove (7). Let $f = (f_1, \dots, f_n)$ and $\theta = (\theta_1, \dots, \theta_n)^T \in \partial\mathbb{B}^n$. Without loss of generality, we assume that f is also h -harmonic on $\partial\mathbb{B}^n$. If we consider the formula (8) for f componentwise and then the partial derivatives with respect to the variables z_k and \bar{z}_k , we see that

$$(f_j(z))_{z_k} = \int_{\partial\mathbb{B}^n} \frac{-(2n-1)(1 - |z|^2)^{2n-2} [\bar{z}_k |\zeta - z|^2 + (1 - |z|^2)(\bar{z}_k - \bar{\zeta}_k)]}{|z - \zeta|^{4n}} f_j(\zeta) d\sigma(\zeta)$$

and

$$(f_j(z))_{\bar{z}_k} = \int_{\partial\mathbb{B}^n} \frac{-(2n-1)(1 - |z|^2)^{2n-2} [z_k |\zeta - z|^2 + (1 - |z|^2)(z_k - \zeta_k)]}{|z - \zeta|^{4n}} f_j(\zeta) d\sigma(\zeta)$$

which hold clearly for each $j, k \in \{1, \dots, n\}$. Now, we introduce

$$\Gamma_{f_j} = \sum_{k=1}^n (f_j(z))_{z_k} \cdot \theta_k + \sum_{k=1}^n (f_j(z))_{\bar{z}_k} \cdot \bar{\theta}_k.$$

Then the classical Cauchy-Schwarz inequality yields

$$\begin{aligned} &\frac{|\Gamma_{f_j}|^2}{(2n-1)^2(1 - |z|^2)^{4n-4}} \\ &= \left| \sum_{k=1}^n \int_{\partial\mathbb{B}^n} \frac{[\bar{z}_k |\zeta - z|^2 + (1 - |z|^2)(\bar{z}_k - \bar{\zeta}_k)] \theta_k}{|z - \zeta|^{4n}} f_j(\zeta) d\sigma(\zeta) \right. \\ &\quad \left. + \sum_{k=1}^n \int_{\partial\mathbb{B}^n} \frac{[z_k |\zeta - z|^2 + (1 - |z|^2)(z_k - \zeta_k)] \bar{\theta}_k}{|z - \zeta|^{4n}} f_j(\zeta) d\sigma(\zeta) \right|^2 \\ &\leq 4 \left[\int_{\partial\mathbb{B}^n} \frac{[|z| |\zeta - z|^2 + (1 - |z|^2) |\zeta - z|] |f_j(\zeta)|}{|z - \zeta|^{4n}} d\sigma(\zeta) \right]^2 \\ &\leq 4 \left[\int_{\partial\mathbb{B}^n} \frac{[|z| |\zeta - z| + (1 - |z|^2)]^2}{|z - \zeta|^{4n-2}} d\sigma(\zeta) \right] \left[\int_{\partial\mathbb{B}^n} \frac{|f_j(\zeta)|^2}{|z - \zeta|^{4n}} d\sigma(\zeta) \right], \end{aligned}$$

whence

$$\begin{aligned}
& \frac{|\Lambda_f|^2}{(2n-1)^2(1-|z|^2)^{4n-4}} \\
&= \frac{\max_{\theta \in \partial \mathbb{B}^n} \left(\sum_{j=1}^n |\Gamma_{f_j}|^2 \right)}{(2n-1)^2(1-|z|^2)^{4n-4}} \\
&\leq 4 \left[\int_{\partial \mathbb{B}^n} \frac{[|z||\zeta - z| + (1-|z|^2)]^2}{|z - \zeta|^{4n-2}} d\sigma(\zeta) \right] \left[\int_{\partial \mathbb{B}^n} \frac{\sum_{j=1}^n |f_j(\zeta)|^2}{|z - \zeta|^{4n}} d\sigma(\zeta) \right] \\
&\leq \frac{4M^2}{(1-|z|)^2(1-|z|^2)^{2n-1}} \left[\int_{\partial \mathbb{B}^n} \frac{(1+|z|)^2}{|z - \zeta|^{4n-2}} d\sigma(\zeta) \right] \\
&\leq \frac{4M^2(1+|z|)^2}{(1-|z|)^2(1-|z|^2)^{2n-1}} \left[\int_{\partial \mathbb{B}^n} \frac{1}{|z - \zeta|^{4n-2}} d\sigma(\zeta) \right] \\
&\leq \frac{4M^2(1+|z|)^2}{(1-|z|)^2(1-|z|^2)^{4n-2}}.
\end{aligned}$$

Hence

$$|\Lambda_f|^2 \leq \frac{4(2n-1)^2 M^2}{(1-|z|)^4},$$

from which the inequality (7) follows. \square

Definition 4. A matrix-valued function $A(z) = (a_{i,j}(z))_{n \times n}$ is called *h-harmonic* if each of its entries $a_{i,j}(z)$ is a *h*-harmonic mapping from an open subset $\Omega \subset \mathbb{C}^n$ into \mathbb{C} .

As an application of Theorem 2, we get

Lemma 3. Suppose that $A(z) = (a_{i,j}(z))_{n \times n}$ is a matrix-valued *h*-harmonic mapping of $\mathbb{B}^n(r)$ such that $A(0) = 0$ and $|A(z)| \leq M$ in $\mathbb{B}^n(r)$. Then

$$|A(z)| \leq M \left[1 - \frac{(r-|z|)^{2n-1}}{(r+|z|)^{2n-1}} \right].$$

Proof. For an arbitrary $\theta = (\theta_1, \dots, \theta_n)^T \in \partial \mathbb{B}^n$, we introduce

$$P_\theta(z) = A(z)\theta = (p_1(z), \dots, p_n(z))$$

and let $F_\theta(\zeta) = P_\theta(r\zeta)$ for $\zeta \in \mathbb{B}^n$. By Theorem 2, we see that

$$\left| F_\theta(\zeta) - \frac{(1-|\zeta|)^{2n-1}}{(1+|\zeta|)^{2n-1}} F_\theta(0) \right| \leq M \left[1 - \frac{(1-|\zeta|)^{2n-1}}{(1+|\zeta|)^{2n-1}} \right], \quad \zeta \in \mathbb{B}^n,$$

which is equivalent to

$$|P_\theta(z)| \leq M \left[1 - \frac{(r-|z|)^{2n-1}}{(r+|z|)^{2n-1}} \right], \quad z \in \mathbb{B}^n(r).$$

The arbitrariness of θ yields the desired inequality. \square

We recall the following result which is crucial for the proof of our next theorem.

Lemma A. ([5, Lemma 1] or [15, Lemma 4]) *Let A be an $n \times n$ complex (real) matrix. Then for $\theta \in \partial \mathbb{B}^n$, the inequality $|A\theta| \geq |\det A| |A|^{1-n}$ holds.*

Theorem 3. *Suppose that $f \in \mathcal{HB}_n(\alpha)$, $f(0) = 0$, $\det J_f(0) = 1$ and $\|f\|_{\mathcal{HB}_n(\alpha)} \leq M$, where M is a positive constant. Then f is univalent in $\mathbb{B}^n(\rho/2)$, where*

$$(9) \quad \rho = \frac{3^\alpha}{(2M)^{2n}(3^\alpha + 4^\alpha)}.$$

Moreover, the range $f(\mathbb{B}^n(\rho/2))$ contains a univalent ball $\mathbb{B}^n(R)$, where

$$R \geq \frac{\rho}{4M^{2n-1}}.$$

Proof. For $\zeta \in \mathbb{B}^n$, let $F(\zeta) = 2f(\frac{1}{2}\zeta)$. Then

$$|F_\zeta(\zeta)| + |F_{\bar{\zeta}}(\zeta)| \leq \frac{M}{\left(1 - \frac{|\zeta|^2}{4}\right)^\alpha} \leq \frac{4^\alpha}{3^\alpha} M$$

which gives

$$|F_\zeta(\zeta) - F_\zeta(0)| \leq |F_\zeta(\zeta)| + |F_\zeta(0)| \leq \left(1 + \frac{4^\alpha}{3^\alpha}\right) M.$$

Lemma 3 implies that

$$\begin{aligned} |F_\zeta(\zeta) - F_\zeta(0)| &\leq \left(1 + \frac{4^\alpha}{3^\alpha}\right) M \left[1 - \frac{(1 - |\zeta|)^{2n-1}}{(1 + |\zeta|)^{2n-1}}\right] \\ &= \frac{2M(3^\alpha + 4^\alpha)}{3^\alpha} \frac{(C_{2n-1}^1|\zeta| + C_{2n-1}^3|\zeta|^3 + \dots + C_{2n-1}^{2n-1}|\zeta|^{2n-1})}{(1 + |\zeta|)^{2n-1}} \\ &\leq \frac{2^{2n-1}(3^\alpha + 4^\alpha)M}{3^\alpha(1 + |\zeta|)^{2n-1}} |\zeta| \\ (10) \quad &\leq \frac{2^{2n-1}(3^\alpha + 4^\alpha)M}{3^\alpha} |\zeta|, \end{aligned}$$

where $C_n^k = \binom{n}{k}$ ($k = 1, 2, \dots, n$) denote the binomial coefficients. Similarly,

$$(11) \quad |F_{\bar{\zeta}}(\zeta) - F_{\bar{\zeta}}(0)| \leq \frac{2^{2n-1}(3^\alpha + 4^\alpha)M}{3^\alpha} |\zeta|.$$

On the other hand, for $\theta \in \partial \mathbb{B}^n$, we infer from (1), (2) and Lemma A that

$$(12) \quad \lambda_F(0) \geq \frac{\det J_F(0)}{\Lambda_F^{2n-1}(0)} \geq \frac{1}{M^{2n-1}}.$$

In order to prove the univalence of F in $\mathbb{B}^n(\rho)$, we choose two distinct points ζ' and ζ'' in $\mathbb{B}^n(\rho)$ with $\zeta' - \zeta'' = |\zeta' - \zeta''|\theta$, and let $[\zeta', \zeta'']$ denote the line segment with endpoints ζ' and ζ'' , where

$$\rho = \frac{3^\alpha}{(2M)^{2n}(3^\alpha + 4^\alpha)}.$$

Set $d\zeta = (d\zeta_1, \dots, d\zeta_n)^T$ and $(d\bar{\zeta} = (d\bar{\zeta}_1, \dots, d\bar{\zeta}_n)^T$. Then we infer from (10), (11) and (12) that

$$\begin{aligned}
|F(\zeta') - F(\zeta'')| &\geq \left| \int_{[\zeta', \zeta'']} F_\zeta(0) d\zeta + F_{\bar{\zeta}}(0) d\bar{\zeta} \right| \\
&\quad - \left| \int_{[\zeta', \zeta'']} (F_\zeta(\zeta) - F_\zeta(0)) d\zeta + (F_{\bar{\zeta}}(\zeta) - F_{\bar{\zeta}}(0)) d\bar{\zeta} \right| \\
&\geq |F_\zeta(0)\theta + F_{\bar{\zeta}}(0)\bar{\theta}| \int_{[\zeta', \zeta'']} |d\zeta| - \frac{2^{2n}(3^\alpha + 4^\alpha)M}{3^\alpha} \int_{[\zeta', \zeta'']} |\zeta| |d\zeta| \\
&> |\zeta' - \zeta''| \left\{ \lambda_F(0) - \frac{2^{2n}(3^\alpha + 4^\alpha)M}{3^\alpha} \rho \right\} \\
&\geq |\zeta' - \zeta''| \left\{ \frac{1}{M^{2n-1}} - \frac{2^{2n}(3^\alpha + 4^\alpha)M}{3^\alpha} \rho \right\} \\
&= 0,
\end{aligned}$$

where $\theta = \frac{d\zeta}{|d\zeta|}$. Thus, F is univalent in $\mathbb{B}^n(\rho)$ which is equivalent to saying that f is univalent in $\mathbb{B}^n(\rho/2)$.

Furthermore, for each z with $|\zeta| = \rho$, we have

$$\begin{aligned}
|F(\zeta) - F(0)| &\geq \left| \int_{[0, \zeta]} F_\zeta(0) d\zeta + F_{\bar{\zeta}}(0) d\bar{\zeta} \right| \\
&\quad - \left| \int_{[0, \zeta]} (F_\zeta(\zeta) - F_\zeta(0)) d\zeta + (F_{\bar{\zeta}}(\zeta) - F_{\bar{\zeta}}(0)) d\bar{\zeta} \right| \\
&\geq \rho \left\{ \frac{1}{M^{2n-1}} - \frac{2^{2n-1}(3^\alpha + 4^\alpha)M\rho}{3^\alpha} \right\} \\
&= \frac{\rho}{2M^{2n-1}} \quad (\text{by (9)})
\end{aligned}$$

showing the range $f(\mathbb{B}^n(\rho/2))$ contains a univalent ball $\mathbb{B}^n(R)$, where $R \geq \rho/(4M^{2n-1})$. The proof of this theorem is complete. \square

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